I would like to see your progress after 1 week of work.

   Topics of this homework: Simulation of the middle ear in the time domain.

   Estimated times are given for solving the problem, but not for writing up the solution, which could take longer than coding up the solution.

   Some constants: $P_0 = 10^5$ [Pa], $\rho c_0 = 407$ [Rayls], $c_0 = 345$ [m/s], $\eta_0 = 1.4$ (adiabatic correction to the speed of sound).

   Grading: 1 point for showing work; 2 points for writeup; 7 points for your solution. Total of 10 points (The HW grade may be rescaled at the end of the course).

Simulation of the middle ear   The middle ear may be treated as an $l = 2.5$ [cm] stub of transmission line, terminated in a series combination of a stiffness, shown as a capacitor C_{al} and damping, shown as a resistor R_c. The experimental details of the cat ear have been extensively explored by Guinan and Peake (1967), and first modeled by Zwislocki (1962, 1957) and much later by Lynch *et al.* (1982). The model shown here is a significant simplification, but adequate for our purpose.[1]
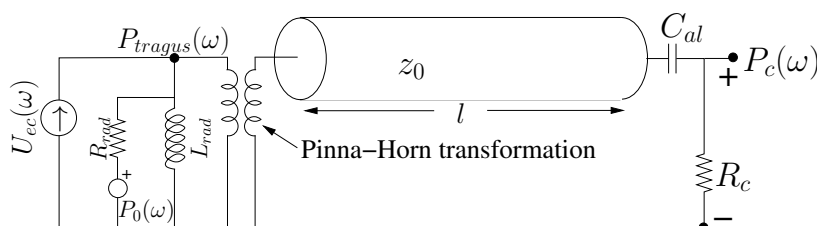

Figure 1:   *Model of the ear canal, terminated by the radiation impedance $Z_{rad}(s)$ at the tragus ($x = 0$), and by the eardrum and cochlea at $x = l$) (i.e., 2.5 [cm]). The horn transformation is the transformer that represents the concha, that converts the area of the canal to the area of the pinna/concha opening. This transformer ratio seems necessary to reduce the effective radiation mass seen by the ear canal, to improve the matching of energy between the canal and free-space. It does this by making the area defining L_{rad} larger by the transformer turns ratio. The effective turns ratio needs to be estimated from measured data (Rosowski et al., 1988, Fig. 10).*

   The diameter of the ear canal is $d_c = 0.75 = 2 * r_c$ cm (area $A_c = \pi r_c^2$). As shown in the figure, the free field sound pressure, defined as $P_0(\omega)$ acts as a source in series with the radiation impedance $Z_{rad}(s, A_{rad})$, composed of a combination of resistance R_{rad} and mass L_{rad}, where

$$R_{rad} = \rho c / A_{rad}, \qquad L_{rad} \equiv r_c \rho / A_{rad}, \tag{1}$$

which represents the local stored energy field. The two impedances are in parallel, thus

$$Z_{rad}(s) = sL_{rad}R_{rad}/(sL_{rad} + R_{rad}) = 1/Y_{rad}(s),$$

where $s = \sigma + j\omega$ is the Laplace complex frequency variable. The *radiation admittance for a sphere* is similar in form (it differs in the area):

$$Y_{rad} = 1/Z_{rad} = \frac{A_c}{sr_c\rho} + \frac{A_c}{\rho c},$$

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[1]An almost identical homework was the basis of the masters thesis of ECE537 student Pierre Parent (Parent and Allen, 2007, 2010).

where r_c is the radius of the sphere and A_{rad} is the *effective area* of the radiation (i.e., $4\pi r^2$ for a sphere and $2\pi r^2$ for a spherical piston mounted on a plate radiator (i.e., a half-sphere)).

If we assume that the pinna (the flap of skin we call our ear) changes the radiation area (i.e., if we model the pinna as a horn), then we can account for this pinna-horn effect as a transformer. The effective area is equivalent to *resonate scattering* characterized by $kr_{rad} = 1$, where $k = \omega/c = 2\pi/\lambda$. Thus the larger the effective area the smaller the impact of the mass reactance L_{rad} on the radiation efficiency. Namely, the size of L_{rad} decreases as the effective area is made larger via the action of the horn transformer, rendering the ear a better collector of sound energy. We can chose the area to match the frequency where the radiation load switches from a mass to a resistance. This frequency was found to be 3 kHz in the cat ear (Rosowski *et al.*, 1988, Fig. 10).

The model: To model the middle ear we first need to simulate the transmission line by using d'Alembert solution of the wave equation. This simulation will operate in the time domain (Kelly and Lochbaum, 1963; Bilbao, 2001).

Create two arrays that represent the forward and backward velocity traveling waves $u_+(x - ct)$ and $u_-(x + ct)$. Assume $c = 345$ m/s. The sampling frequency $F_s = 1/T_s$ must be high enough to get several samples over the length of the ear canal.

Furthermore, if we wish to listen to any signals processed by our ear canal, we need to make F_s a multiple of the audio codec sampling frequency. Common frequencies are 24 and 96 [kHz].

Thus if we assume 7 samples then we may take $F_s = 7c_0/l = 96.04$ [kHz]. This would allow for a 24 [kHz] codec by simply down-sampling by 4.

The boundary conditions at each end of the line will be realized via reflection coefficients $R(L, s)$ at the cochlear end and $R(0, s)$ at the input. Initially assume that P_0 is zero, and drive the middle ear with a velocity source U_{ec} at the tragus.

On the left we terminate the ear in a radiation impedance

$$Z_{rad}(s) = \frac{sL_{rad}R_{rad}}{sL_{rad} + R_{rad}}.$$

At the cochlear end we terminate the line with an impedance

$$Z_c = R_c + 1/sC_{al},$$

where R_c is the cochlear impedance and C_{al} is the stiffness of the annular ligament, which is the ligament that holds the stapes in the oval window. The cochlear resistance (R_c) is assumed to be twice the characteristic impedance of the ear canal.

Each impedance may be converted into a reflection coefficient, defined as

$$R(s) \equiv \frac{U_-}{U_+} = \frac{Z(s) - z_0}{Z(s) + z_0},$$

where $z_0 = \rho c/A_c$ is the characteristic impedance of the ear canal, having area A_c. For example, the reflection coefficient at $x = l$ is the transfer function between the wave reflected $U_-(L, s)$ over the incident wave $U_+(L, s)$, as defined above. The two reflection coefficients are a function of complex frequency s.

Use the bilinear z transform to convert $R(s)$ into a discrete time IIR filter having numerator polynomial $N(z)$ (zeros) and denominator polynomial $D(z)$ (poles), so that the reflectance filters (e.g., $R(0, s)$ and $R(L, s)$) are implemented in the time domain. I recommend that you implement the filter manually rather than use the `filter()` command. [If you chose to use the `filter()` command, *you must save state* between calls. See `help filter` for the details on how to do this, and *carefully verify that it is working.*]

Finally is the question of the boundary conditions. This is where things are a little tricky. These are dealt with by defining the end reflection coefficients

$$R(L, s) \equiv \frac{U_-(L, s)}{U_+(L, s)}$$

and

$$R(0, s) \equiv \frac{U_+(0, s)}{U_-(0, s)}.$$

In the frequency domain we can find

$$U_-(L, s) = R(L, s)U_+(L, s)$$

and

$$U_+(0, s) = R(0, s)U_-(0, s).$$

These relations give the junction reflected wave in terms of the incident wave. For example, at $x = l$ the incident wave is U_+ and the reflected wave is U_-. At $x = 0$ the signs must change. Since this relation is in the frequency domain, it represents a convolution in the time domain. Since we wish to run the middle ear model in the time domain, we must design the filter represented by these reflection coefficients. To do this we may use the bilinear z transform.

Each reflection coefficient R is given by

$$R(s) = \frac{Z(s) - z_0}{Z(s) + z_0}.$$

I derived this formula in class and showed that $Z(s)$ is the load impedance and $z_0 = \rho c/A$ is the *characteristic impedance* of the transmission line. Taking the case of $x = l$ we find

$$R(L, s) = \frac{R_c + 1/sC_{al} - z_0}{R_c + 1/sC_{al} + z_0}.$$

When using the bilinear z transform one substitutes $2F_s\frac{z-1}{z+1}$ for s. Try `doc bilinear` from Matlab©.

Show your work. Check all the filters and all the components carefully.

Show your work. For example, show the impulse response for each of these filters: $r(0, t) \leftrightarrow R(0, s)$ and $r(L, t) \leftrightarrow R(L, s)$. Test your time domain transmission line impulse response by setting the reflection coefficients to 1, and then compute the impulse response of $p_c(L, t)$ with an impulse at the input (e.g., find $p_{ec}(0, t) = \delta(t)$). Show your work. Write this up with a full description of what you did, so that I can easily understand how you got your results. I will grade this homework partially on the readability of the presentation.

1. To-do:

   (a) Find z_0, the characteristic impedance of the ear canal. (1 min)

   (b) Assume that the cochlear load resistor R_c is equal to twice the characteristic impedance of the ear canal. Find C_{al}, by assuming that the impedance of this element ($Z_{al} \equiv 1/sC_{al}$) is equal to the cochlear impedance R_c at a frequency of 0.800 kHz. Show all your work. (1 min)

   (c) Determine the radiation impedance looking out the ear canal, and compute the poles and zeros. For an example of the frequency response of this impedance look at Rosowski *et al.* (1988, Fig. 10) provided on the class web site.

3

(d) The simulation is to be done in the time domain. Define two arrays in Matlab© called $u_+(x - ct)$ and $u_-(x + ct)$, which represent the velocity of the waves in the ear canal, traveling in the two directions. At each time step, the data in the first array is shifted to the right, while the second is shifted to the left. The distance between two points corresponds in the array is $dx = cT$ where sampling rate $T = 1/F_s$. Choose the sampling rate F_s such that there are at least 5 points along the length of the ear canal (you can modify the length of the canal slightly to make the samples come out at the ends).

(e) Find the formula for the input impedance $Z(0, s)$ of the middle ear at the entrance of the ear canal, when the cochlea is "blocked" ($Z_c = \infty$ or $R(L, s) = 1$)? (Hint, this has a simple answer that you can easily derive).

(f) Find the two reflection coefficients and plot their magnitude frequency response.

(g) What are the coefficients of the time domain filters that implement the reflection coefficients?

(Hint 1: Use Matlab's `bilinear()` command. Hint 2: here is the answer to 1 decimal place, to help you debug up to this point. I've used a sampling rate of $\approx$70 kHz to compute this: $R(0, z^{-1}) = -(.5 + .08z^{-1})/(1 - .4z^{-1})$ and $R(L, z^{-1}) = (0.3 - 0.3z^{-1})/(1 - 0.95z^{-1})$.)

(h) What are the poles and zeros in s and in z^{-1} planes?

(i) Compute and plot the input impedance (numerically derived at $x = 0$), using your Matlab model at the point labeled $P_{ec}(\omega)$ (i.e., at $x = 0$). Use log-frequency [Hz] and dB for the magnitude. Use z_0 as the dB reference (e.g., plot $20 \log_{10}(|Z_c(0, f)|/z_0)$). Dont forget that $Z \equiv P/U$ where $p = (u^+ + u^-) * z_0$ and $u \equiv u^+ - u^-$. Compute the time sequence for p and u and then FFT these and take the ratio. Be sure that each function has settled to zero or else you will alias the answer. I am not asking for the transmission line model, I'm looking for the numerical solution.

(j) Using the time domain model, compute the *tragus* transfer function of the middle ear, defined as the ratio of the cochlear pressure to the ear canal pressure $P_c(\omega)/P_{ec}(\omega)$. Pressure P_{ec} is more precisely P_{tragus}.

(k) Compute the free field transfer function of the middle ear, defined as the ratio of the cochlear pressure $P_c(\omega)/P_0(\omega)$. I suggest you work backward, and find the pressure P_0, the assumed source, required to give the given source U_{ec}.

Note that this question may be just too hard, and take to long, so do this problem last (or just skip it if you don't have the time to deal with it). (30-60 min)

(l) A 16 electrical ohm earphone delivers 120 dB SPL into the ear canal with 1 volt RMS input at 1 kHz. Using your middle ear model, calculate the efficiency (Acoustic/Electrical power ratio, in %) of the earphone, at 1 kHz.

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